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Chebyshev Approximation by Exponential Differences

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Let $\beta > 0$. Define the approximating function

$$F(A, x) = \exp(a_1 x) - \exp(a_2 x).$$

We have for all A,

$$F(A, 0) = 0. \qquad (*)$$

Consider Chebyshev approximation of continuous f by F on the closed interval $[0, \beta]$. Because of (*) we only consider f with $f(0) = 0$.

DEFINITION. The *degree* $d(A)$ of F at A is two if $a_1 \neq a_2$ and one if $a_1 = a_2$.

LEMMA. *If $A \neq B$, then $F(A, \cdot) - F(B, \cdot)$ has at most $d(A) - 1$ zeros on $(0, \infty)$.*

The case of $d(A) = 1$ is obvious. All cases of $d(A) = 2$ can be put (by renaming if necessary) into the form of the following proposition.

PROPOSITION. *Let $f(x) = \exp(a_1 x) - \exp(a_2 x) - \exp(b_1 x) + \exp(b_2 x) \not\equiv 0$ and $b_2 \geqslant a_1$, $b_1 \geqslant a_2$. Then f has at most one zero on $(0, \infty)$ if*

$$(a_1 + b_2 - a_2 - b_1)(b_2 - b_1) < 0$$

and f has no zero on $(0, \infty)$, otherwise.

Proof. If $f \neq 0$ and one of the cases $a_1 = a_2$, $a_1 = b_1$, $a_2 = b_2$ and $b_1 = b_2$ occurs then f has no zero on $(0, \infty)$ [1]. In what follows, hence, we assume that $a_1 \neq a_2$, $a_1 \neq b_1$, $a_2 \neq b_2$ and $b_1 \neq b_2$.

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Without loss of generality we assume that

(I) $a_1 + b_2 \geqslant a_2 + b_1$, otherwise we consider $-f$ instead of f;

(II) $a_1, a_2, b_1, b_2 > 0$, otherwise we consider $\exp(rx) \cdot f$ instead of f, where $r = -\min(a_1, a_2, b_1, b_2) + 1$.

At first we note that f has at most two zeros on $(0, \infty)$ by [1] because 0 is its zero. We discuss the following cases.

(a) $a_1 + b_2 > a_2 + b_1$ and $b_2 < b_1$. When $x > 0$ is very small we have

$$\begin{aligned} f &\approx (1 + a_1 x) - (1 + a_2 x) - (1 + b_1 x) + (1 + b_2 x) \\ &= (a_1 + b_2 - a_2 - b_1)x > 0. \end{aligned}$$

But $b_1 > a_2$ and $b_1 > b_2$ implies that $f(x) < 0$ for large x. So the number of zero of f on $(0, \infty)$ is odd and, therefore, is one.

(b) $a_1 + b_2 \geqslant a_2 + b_1$ and $b_2 > b_1$. Note that for $n \geqslant 2$ we have

$$\begin{aligned} b_2^n - b_1^n &= (b_2 - b_1)(b_2^{n-1} + b_2^{n-2} b_1 + \cdots + b_1^{n-1}) \\ &> (a_2 - a_1)(a_2^{n-1} + a_2^{n-2} a_1 + \cdots + a_1^{n-1}) \\ &= a_2^n - a_1^n. \end{aligned}$$

Hence

$$a_1^n + b_2^n > a_2^n + b_1^n, \qquad n \geqslant 2. \tag{1}$$

Multiplying the inequality $a_1 + b_2 \geqslant a_2 + b_1$ by $x > 0$ and (1) by $x^n/n!$ for $n \geqslant 2$, and then adding these inequalities we have

$$\exp(a_1 x) + \exp(b_2 x) > \exp(a_2 x) + \exp(b_1 x), \qquad x > 0.$$

Thus $f(x) > 0$ is always valid on $(0, \infty)$ and f has no zero at all on $(0, \infty)$.

(c) $a_1 + b_2 = a_2 + b_1$ and $b_2 < b_1$. In this case we have by the same argument

$$b_2^n - b_1^n < a_2^n - a_1^n, \qquad n \geqslant 2$$

and

$$\exp(a_1 x) + \exp(b_2 x) < \exp(a_2 x) + \exp(b_1 x), \qquad x > 0.$$

So $f(x) < 0$ on $(0, \infty)$ and still f has no zero.

DEFINITION.

$$F_k(A, x) = \frac{\partial}{\partial a_k} F(A, x), \qquad D(A, B, x) = \sum_{k=1}^{2} b_k F_k(A, x).$$

We have

$$D(A, B, x) = x[b_1 \exp(a_1 x) - b_2 \exp(a_2 x)].$$

As one or two exponentials are Chebyshev sets [1], we have

LEMMA. *$\{D(A, B, \cdot)$: $B \in E_2\}$ is a Haar subspace of dimension $d(A)$ on* $(0, \infty)$.

THEOREM. *$F(A, \cdot)$ is best to f if and only if $f - F(A, \cdot)$ alternates $d(A)$ times on $[0, \beta]$. A best approximation is unique.*

This is a consequence of the lemmas and arguments similar to those of [2].

Approximation on $[0, \infty]$ is also of interest. Let $a_1, a_2 < 0$ and $f(\infty) = \infty$, then exactly the same theory holds (use [2] with both endpoints null points).

The problem could be generalized in two ways. First, differences of other functions could be used. Second, more than one difference could be used. Either of these makes the problem very difficult.

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